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The nS(G)-Autonilpotency of Groups

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Abstract

The various series, definitions such as nilpotency and solubility of groups and all kinds of automorphisms have been the idea of many researchers' articles. In this paper, we first study autonilpotent group and their generalizations. Then we give a new definition for nS(G)-autonilpotency and discuss some properties of this concept.

Keywords: IA-group, IA-central subgroup, autonilpotent group, S(G)-autonilpotent groups, nS(G)-autonilpotent groups.

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1. Introduction

Let G be a group. Let us denote by $Z(G)$, G', and $Aut(G)$, respectively, the centre, the commutator subgroup and the full automorphism group. Let $H \leqslant G$, then

$$C_{Aut(G)}(H) = \{\alpha \in Aut(G) \mid \alpha(h) = h,\ \forall\, h \in H\}.$$

Bachmuth [1] in 1965 defined an IA-automorphism of a group G as

$$IA(G) = \left\{\alpha \in Aut(G) \;\middle|\; g^{-1}\alpha(g) = [g, \alpha] \in G',\ \forall\, g \in G\right\}.$$

Hegarty [4] in 1994 introduced the absolute center

$$L(G) = \left\{g \in G \;\middle|\; g^{-1}\alpha(g) = 1,\ \forall\, \alpha \in Aut(G)\right\}.$$

On the similar lines, Ghumde and Ghate [3] in 2015 introduced the IA-central subgroup

$$S(G) = \left\{g \in G \mid g^{-1}\alpha(g) = 1,\ \alpha \in IA(G)\right\}.$$

For any group G, $L(G) \trianglelefteq S(G) \trianglelefteq Z(G)$.

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2. Main results

Parvaneh and Moghaddam [7] in 2010 introduced the concept of autonilpotent groups. They defined the upper autocentral series of G as

$$\langle 1\rangle = L_0(G) \subseteq L_1(G) = L(G) \subseteq L_2(G) \subseteq \cdots \subseteq L_n(G) \subseteq \cdots$$

where

$$\frac{L_n(G)}{L_{n-1}(G)} = L\Big(\frac{G}{L_{n-1}(G)}\Big), \qquad \text{for } n \geqslant 2$$

and $L_n(G)$ is the nth-absolute centre of G. Also, they called a group autonilpotent of class at most c if $L_c(G) = G$, for some positive integer c.

For each natural number i and n, we [2] defined

$$L_i^n(G) = \{g \in G \mid [g, \alpha_1^n, \alpha_2^n, \ldots, \alpha_i^n] = 1,\ \forall\, \alpha_1, \alpha_2, \ldots, \alpha_i \in Aut(G)\}.$$

Also, we called a group G to be an n-autonilpotent group of class at most c if there exists some positive integer c such that $L_c^n(G) = G$.

Thereafter, we define the IA-central series of G in the following way:

$$\langle 1\rangle = S_0(G) \subseteq S_1(G) = S(G) \subseteq S_2(G) \subseteq \cdots \subseteq S_i(G) \subseteq \cdots$$

where

$$S_i(G) = \{g \in G \mid [g, \alpha_1, \alpha_2, \ldots, \alpha_i] = 1,\ \forall\, \alpha_1, \alpha_2, \ldots, \alpha_i \in IA(G)\}, \quad i \geqslant 1.$$

A group G is called S(G)-autonilpotent(or IA-nilpotent) group of class at most c if $S_c(G) = G$, for some positive integer number c.

In this section, we generalize the concept of S(G)-autonilpotency and represent their properties.

2.1. Preliminary Results

Definition 2.1. For each positive integer i and n, we define

$$S_i^n(G) = \{g \in G \mid [g, \alpha_1^n, \alpha_2^n, \ldots, \alpha_i^n] = 1,\ \forall\, \alpha_1, \alpha_2, \ldots, \alpha_i \in IA(G)\}.$$

Definition 2.2. A group G is called nS(G)-autonilpotent group of class at most c if $S_c^n(G) = G$, for some positive integer c.

Example 2.3. For an abelian group G, we know that IA(G) is trivial, so $S_i^n(G) = G$, for every positive integer i. Therefore, abelian groups are nS(G)-autonilpotent.

Remark 2.4. Clearly, for a group G and every positive integer i, $L_i^n(G) \leqslant S_i^n(G)$. Thus, n-autonilpotent groups are nS(G)-autonilpotent groups, but the converse of this result is not generally valid.
For example, $\mathbb{Z}_3$ is nS(G)-autonilpotent, but is not n-autonilpotent.

Proposition 2.5. Let G be any group, then for each $g \in G$ we have

$$g \in S_i^n(G) \iff [g, \alpha] \in S_i^{n-1}(G), \quad \forall\, \alpha \in IA(G).$$

Proof. Due to $S_i^n(G)$ definition and by inductive on i, the lemma is proved. □

As an immediate consequence of the above proposition, we have the following corollary.

Corollary 2.6. For each $g \in G$, we have

$$g \in S_i^n(G) \iff [S_i^{n-1}(G), IA(G)] = 1.$$

Lemma 2.7. Let G be a non-trivial nS(G)-autonilpotent group, then $S(G) \neq \langle 1 \rangle$.

Proof. By the hypothesis, there exist a positive integer i such that $S_i^n(G) = G$. We assume by way of contradiction that $S(G) = \langle 1 \rangle$, then according to $S_i^n(G)$ definition and by proposition 2.5, $S_2^n(G) = \langle 1 \rangle$. Thus, we have $S_i^n(G) = \langle 1 \rangle$, for every positive integer i, contrary to the assumption. Hence $S(G) \neq \langle 1 \rangle$. □

Theorem 2.8. Let G be a group and H_1 and H_2 be two characteristic subgroups of it. If G is the direct product of H_1 and H_2, then for all $i \geqslant 1$,

$$S_i^n(H_1 \times H_2) = S_i^n(H_1) \times S_i^n(H_2).$$

Proof. The Theorem holds by induction on i. □

Corollary 2.9. If H_1 and H_2 be two finite groups such that $(|H_1|, |H_2|) = 1$, then

$$S_i^n(H_1 \times H_2) = S_i^n(H_1) \times S_i^n(H_2).$$

Corollary 2.10. Let G be a group and H_1 and H_2 be two characteristic subgroups of it. If G is the direct product of H_1 and H_2 such that one of them is not nS(G)-autonilpotent, then so is not G.

Corollary 2.11. If G_1, G_2, ..., G_k are nS(G)-autonilpotent groups with coprime orders, then so is

$$G_1 \times G_2 \times \cdots \times G_k.$$

2.2. When $S_i^n(G) \neq \langle 1 \rangle$?

Now, we study the conditions in which $S_i^n(G)$ is non-trivial. We saw that for abelian groups $S_i^n(G) = G$. Therefore, in the following, we consider non-abelian groups.

Theorem 2.12. Let G be a group and $H \leqslant G$, then $H \leqslant S_i^n(G)$ if one of the following conditions holds:

1) $Aut(G) = C_{Aut(G)}(H)$.
2) G be a finite group and H be a characteristic subgroup of prime order p such that p be the smallest prime divisor of $|Aut(G)|$.
3) H be a cyclic characteristic subgroup of G and Aut(G) be a perfect group.

Proof. Given that $L(G) \leqslant S(G) \leqslant S_i^n(G)$, the proof easily follow from [6] lemma 2.4(iv), corollary 3.5 and 3.7, respectively. □

Theorem 2.13. Let G be a group, Aut(G) be a finite p-group and H be a finite characteristic subgroup of G such that $p \big| |H|$, then $H \cap S_i^n(G) \neq \langle 1 \rangle$.

Proof. Because H is a characteristic subgroup of G, then this equivalence relation yields a partition of H and each cell in the partition arising from an equivalence relation is an equivalence class. According to lemma 2.5 [6], there is $1 \neq h_0 \in H$ element such that the equivalence class is of order 1. So we have $\alpha(h_0) = h_0$, for every $\alpha \in Aut(G)$. Thus $1 \neq h_0 \in S(G) \cap H$ and this completes the proof. □

Corollary 2.14. If G is a finite group such that Aut(G) is a p-group, then $S_i^n(G) \neq \langle 1 \rangle$.

Theorem 2.15 (MacHale[5]). Let G be a finite group such that Aut(G) is nilpotent. If G is not cyclic of odd order, then G contains a non-trivial element which is left fixed by every automorphism of G.

Corollary 2.16. Let G be a finite group such that Aut(G) is nilpotent, then $S_i^n(G) \neq \langle 1 \rangle$.

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